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Is it possible to reconstruct the freeze-out duration of heavy-ion collisions using tomography?

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Abstract

We investigate what conditions allow us to extract the relative distribution of freeze-out space and time points in an arbitrary reference frame using tomography and source imaging. The source function may be extracted from the two-particle correlation function measured in heavy-ion collisions using imaging techniques. This imaged source function is related to the relative distribution of freeze-out space and time points through a generalization of the Radon transform found in tomography. Using tomography, the imaged source function may be converted into the relative freeze-out distribution in the frame of interest. We describe how the tomography may be performed in practice.

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We intend to create the quark-gluon plasma in central nuclear collisions at the Relativistic Heavy-Ion Collider (RHIC). The quark-gluon plasma is predicted to have a long lifetime [1,2] which should lead to large time separations between the emission (or freeze-out) times of pairs of like particles. As two-pion correlations are sensitive to this time separation, they should be a useful tool for studying the plasma [2,3]. The signal of a long lifetime would be anomalously large correlation radii, particularly in the longitudinal and outward directions (in a analysis done in the Bertsch-Pratt parameterization [4]). The lifetime information is encoded in these directions in a non-trivial manner which is currently accessed only using model parameterizations of two-pion correlation functions [2]. A long lifetime should also effect other like-pair correlations, such as two-proton or two-kaon correlations [5–8]. However as in the case of protons, final state interactions often obscure this information.

Recently it was shown that one can perform model-independent extractions of the *entire* source function $S(\vec{r})$ from two-particle correlations, not just its radii, using imaging techniques [9–11]. Furthermore, one can do this even with relatively complicated final-state interactions (e.g. protons) and without making any *a priori* assumptions about the source geometry or lifetime, etc. First results from the application of imaging to the proton, pion, kaon, and IMF correlation data can be found in Refs. [9–12]. These results are both intriguing and nearly as hard to interpret as the original correlation functions: the reconstructed sources are found in the pair Center-of-Mass (CM) frame and the time dependence is folded into the sources in a non-trivial way.

These two ambiguities are intimately tied together: the time direction is folded into the source function by a line integral in the direction of the boost from the system (or lab) frame to the CM frame. As we show in Eq. (9), this line integral is a generalized Radon transform of the relative distribution of space-time emission points. If we construct the correlation function from pairs with a fixed rapidity and transverse momentum, then we know the boost from the system frame to the pair CM frame. Therefore, it should be possible to use tomography to reconstruct the relative distribution of space-time emission points from the imaged sources and we give an explicit inversion formula in Eq. (13). There is one caveat: for the reconstruction to be unique, we must require that the probability for emitting a pair with a certain relative separation be independent of the total pair momentum. In what follows, we will refer to this condition as the momentum averaging approximation. The experimental signal for this condition to hold would be a sideward radius parameter that does not depend on the total pair momentum and this condition seems to be fulfilled at the CERN-SPS [13]. This analysis may be useful in the correlation function analysis of other systems, such as fragmenting hadronic strings or relativistic atomic collisions.

The outline of this letter is as follows. First, we detail the imaging of the source function as a function of pair momentum and show that the imaged source function is a generalized Radon transform of the full two-particle source. Second, we study the reconstruction of the relative space-time distribution of emission points from the imaged source. Finally, we outline how to perform this reconstruction in a practical manner.

We begin with a discussion of the correlation function and its relation to the source function. For concreteness, we consider like-pairs from central relativistic nucleus-nucleus reactions. We are eventually interested in quantities in the lab frame, namely the center-of-mass frame of the colliding nuclear system. The pairs we consider can be protons, pions, etc., as the formalism works for any like-pairs. The like-pair correlation function in an arbitrary frame is the following ratio of invariant spectra:

$$C(P,q) = E_1 E_2 \frac{dN^{\rm true}}{d\vec{p}_1 d\vec{p}_2} \Bigg/ E_1 E_2 \frac{dN^{\rm mixed}}{d\vec{p}_1 d\vec{p}_2} \,. \tag{1}$$

Here, $dN^{\rm true}/d\vec{p}_1 d\vec{p}_2$ is the two-particle spectrum of pairs from the same event (averaged over many events) and $dN^{\rm mixed}/d\vec{p}_1 d\vec{p}_2$ is the two-particle spectrum constructed from pairs in different events. In this expression, $\vec{p}_1$ and $\vec{p}_2$ are the three-momentum of each of the particles and E_1 and E_2 are the on-shell energies. For simplicity, we work in side–out–long coordinates (a.k.a. Bertsch-Pratt [4]) coordinates where the longitudinal axis is along the beam line, the outward axis is along the component of $\vec{P}$ perpendicular to the longitudinal direction, and the sideward axis is perpendicular to the other two directions. In these coordinates, the total four-momentum of the pair is $P = p_1 + p_2 = (E, P_L, P_O, 0)$ and the relative four-momentum is $q = \frac{1}{2}(p_1 - p_2) = (q_0, q_L, q_O, q_S)$.

In the Koonin-Pratt formalism, the correlation in Eq. (1) is related to the normalized single-particle sources through [5,9,10,14]:

$$C(P,q) = \int d^4r\, |\Phi^{(-)\rm rel}(r,q)|^2 \int d^4R\, \tilde{D}(R + r/2, P/2 + q)\tilde{D}(R - r/2, P/2 - q) \tag{2}$$

Here $\Phi^{(-)\rm rel}(r,q)$ is the pair relative wavefunction of the emitted pair and it includes all final-state interactions between the pair as well as (anti-)symmetrization due to statistics. Also $\tilde{D}(r,p)$ is the normalized single-particle source and it denotes the probability of emitting an on-shell particle with four-momentum p at position $\vec{r}$ at time t. In terms of the emission rate, $\tilde{D}(r,p)$ is:

$$\tilde{D}(r,p) = \frac{E\, d^7N}{d^3p\ d^4r} \Bigg/ \frac{E\, d^3N}{d^3p} \tag{3}$$

where the emission rate is the number of particles frozen out per unit time, per unit volume, per unit volume of invariant momentum. Note that the normalized single-particle source transforms as a four-scalar.

We may measure the correlation function and write Eq. (2) in whatever frame we wish. In the pair center-of-mass (CM) frame Eq. (2) simplifies considerably:

$$C_{\vec{P}}(\vec{q}\,') = \int d^4r'\, |\Phi^{(-)\rm rel}(\vec{r}\,', \vec{q}\,')|^2 \int d^4R'\, \tilde{D}_{\vec{P}}(R' + r'/2, \vec{q}\,')\tilde{D}_{\vec{P}}(R' - r'/2, -\vec{q}\,'). \tag{4}$$

The subscript $\vec{P}$ refers to the momentum of the boost from some other frame (e.g. the lab frame) to the pair CM frame and the primed coordinates are pair CM coordinates. Note that in the pair CM frame $\vec{P}' = 0$ and $E' = 2m$. The reader should also note that *each pair has a different $\vec{P}$ so it is boosted to a different pair CM frame.* Furthermore, because of our choice of coordinates, this boost is confined to the longitudinal and outward directions. In order to remove the t' dependence of the wavefunction from Eq. (4), we have taken advantage of the facts that the particles are on-shell, giving $q_0 = \vec{\beta} \cdot \vec{q}$, and that our wavefunction is a function of the Lorentz scalar $q \cdot r = (t\vec{\beta} - \vec{r}\,) \cdot \vec{q} = -\vec{r}\,' \cdot \vec{q}\,'$.

We now make two crucial assumptions: the *smoothness approximation* and a *momentum averaging approximation.* For the smoothness assumption, we assume that the normalized single-particle sources have only a weak dependence on $\vec{q'}$. This is justified because the single-particle source varies weakly in $\vec{r'}$ as a function of $\vec{q'}$ while the wavefunction oscillates rapidly in $\vec{r'}$ for large $\vec{q'}$ [14,15]. Thus, at low $\vec{q'}$ the single-particle source is approximately independent of $\vec{q'}$ while at large $\vec{q'}$ the integral of the source function and the wavefunction average to zero. This approximation allows us to define the source function [9,10]:

$$S_{\vec{P}}(\vec{r'}) = \int dt' \int d^4R' \, \tilde{D}_{\vec{P}}(R' + r'/2, \vec{q'} \approx 0)\tilde{D}_{\vec{P}}(R' - r'/2, \vec{q'} \approx 0). \tag{5}$$

$S_{\vec{P}}(\vec{r'})$ is the probability of creating a pair a distance $\vec{r'}$ apart in the pair CM frame. In terms of the source function, Eq. (4) becomes

$$C_{\vec{P}}(\vec{q'}) = \int d^3r' \, |\Phi^{(-)\mathrm{rel}}(\vec{r'}, \vec{q'})|^2 S_{\vec{P}}(\vec{r'}). \tag{6}$$

Since the source function in Eq. (6) is independent of the relative pair momentum, we can uniquely invert $C_{\vec{P}}(\vec{q'})$ to obtain $S_{\vec{P}}(\vec{r'})$ using the imaging techniques of Refs. [9,10]. Therefore, we can dispense with the correlation function and work directly with the imaged source function.

For reasons that will become clear momentarily, let us define the two-particle source as the probability of creating a pair of particles space-time distance of $r = (t, \vec{r})$ apart in the lab frame. We may write the the two-particle source $\mathcal{S}(r, P)$ in terms of the normalized single-particle sources as

$$\begin{aligned} \mathcal{S}(r, P) &= \int d^4R \, \tilde{D}(R + r/2, P/2)\tilde{D}(R - r/2, P/2) \\ &= \int d^4R' \, \tilde{D}_{\vec{P}}(R' + r'/2, \vec{q'} \approx 0)\tilde{D}_{\vec{P}}(R' - r'/2, \vec{q'} \approx 0). \end{aligned} \tag{7}$$

The second line in this equation is just the first line rewritten in the pair CM frame. The space-time displacement r' in the second line should be understood as a function of the boost velocity and space-time displacement r in the original frame. Here we have dropped the q dependence in both the lab frame (as in the lab frame $P \gg q$) and in the pair CM frame (in the smoothing approximation). Comparing the definition of the source function (4) with the second line of Eq. (7), we see

$$S_{\vec{P}}(\vec{r'}) = \int dt' \, \mathcal{S}(r, P). \tag{8}$$

Thus, the imaged source is a line integral of the two-particle source along some boosted direction given by the pair total momentum, $\vec{P}$.

In the momentum averaging approximation, we replace $\mathcal{S}(r, P)$ in Eq. (8) with $\tilde{\mathcal{S}}(r)$. Here the $\tilde{\mathcal{S}}(r)$ has the interpretation as the relative distribution of space-time emission points, i.e. it gives the probability density for emitting any pair from space-time points separated by a displacement r in the lab frame. In terms of this distribution, we write Eq. (8) as

$$S_{\vec{P}}(\vec{r'}) = \int dt' \, \tilde{\mathcal{S}}(r). \tag{9}$$

Here all of the $\vec{P}$ dependence of the source function is attributed to the boost from the lab frame to the pair CM frame. This transform may be regarded as a generalized Radon transform as it involves boosts instead of rotations. In the conventional Radon transform [17–19], one integrates a function $f(x, y)$ along a direction rotated at an angle θ, e.g.

$$f_\theta(x') = \int dy'\, f(x,y) \tag{10}$$

where $y' = x\sin\theta + y\cos\theta$ and $x' = x\cos\theta - y\sin\theta$. In Eq. (9), we are integrating along a time direction in a boosted frame. Thus, t' in Eq. (9) plays the role of x in the conventional Radon transform and $\vec{P}$ (or equivalently the boost velocity) plays the role of the rotation angle θ.

The momentum averaging approximation is justified for the following reasons. First, we could just *define* the relative distribution of emission points through Eq. (9). In this case, we will be folding any P dependence (such as from collective motion) into the time direction. Alternatively, we may think of $\tilde{\mathcal{S}}(r)$ as $\mathcal{S}(r,P)$, but averaged over P. This is the viewpoint we adopt in this letter. Finally, in some systems or situations the two particle source only has a weak dependence on P and we may neglect this dependence. In a heavy-ion reaction, we can check this approximation by measuring the (e.g. pion) correlation function as a function of total pair momentum. Then we must examine the dependence of the correlation in the sideward and longitudinal/outward directions. If the dependence of the correlation in the sideward direction is relatively flat as a function of the total pair momentum, the two particle source does not have a P dependence. We will illustrate this below in a simple Gaussian model. We comment that the sideward pion radius parameter measured by the CERN-SPS experiment NA49 [13] is relatively flat as a function of pair rapidity and transverse momentum so the momentum averaging approximation seems justified. Strictly speaking, comparing the outward and sideward direction tells us that there is no P dependence of $\mathcal{S}(r,P)$ in the sideward/outward directions, but tells us nothing about the P dependence in the longitudinal direction. We postpone the detailed investigation of this approximation (especially in the presence of collective motion) for a future article [20].

Let us now illustrate how the transform in Eq. (9) works on a model two-particle source with time evolution, but no dependence on the total pair momentum. Assume that we have spherical stationary two-particle source with radius R and time duration τ in the lab frame:

$$\mathcal{S}(t,\vec{r}) = \mathcal{S}_0 \exp\left(-\frac{t^2}{2\tau^2} - \frac{\vec{r}^2}{2R^2}\right). \tag{11}$$

Since $\mathcal{S}(t,\vec{r})$ denotes the probability density for producing the pair with a space-time separation $(t,\vec{r})$, the normalization constant is $\mathcal{S}_0 = (4\pi^2\tau R^3)^{-1}$. The boost from the lab to the pair CM frame is characterized by the boost velocity $\vec{\beta}$, which may be written in terms of the total pair momentum giving $\vec{\beta} = \vec{P}/E \equiv (\beta_L, \beta_O, 0)$. The coordinates transform from the pair CM frame to the system frame via $\vec{r} = \vec{r'} + \frac{\gamma-1}{\beta^2}(\vec{\beta}\cdot\vec{r'})\vec{\beta} + \gamma\vec{\beta}t'$ and $t = \gamma(t' + \vec{\beta}\cdot\vec{r'})$ with $\gamma = 1/\sqrt{1-\vec{\beta}^2}$. Integrating over the CM time to find the imaged source function, we find:

$$S_{\vec{\beta}}(\vec{r'}) = \mathcal{S}_0 \frac{\sqrt{2\pi}}{\gamma}\sqrt{\frac{\tau^2 R^2}{R^2+\beta^2\tau^2}} \exp\left(-\frac{1}{2R^2}\left(\vec{r'}^2 - (\vec{\beta}\cdot\vec{r'})^2 \frac{R^2+\tau^2}{R^2+\beta^2\tau^2}\right)\right). \tag{12}$$

This source function is also Gaussian and we immediately see how the radius parameters are modified by the boost. First, since $\vec{\beta}$ has no component in the sideward direction, the Gaussian radius in the sideward direction is independent of $\vec{\beta}$ and is equal to the radius of the original two-particle source, $R_S = R$. We remind the reader that a flat sideward radius parameter is a signal of the validity of the momentum averaging approximation. Second, both the outward and longitudinal radius parameters become $R_{O/L} = R/\sqrt{1-\beta_{O/L}^2(R^2+\tau^2)/(R^2+\beta^2\tau^2)}$. So for large emission durations, we find anomalously large radius parameters in the outward and longitudinal directions. Finally, we find a long/out cross term $R_{OL}^2 = -R^2/\beta_L\beta_O\sqrt{(R^2+\tau^2)/(R^2+\beta^2\tau^2)}$.

Now we can invert Eq. (9) and obtain the inverse in the Bertsch-Pratt coordinates:

$$\begin{aligned}&\tilde{\mathcal{S}}(t, r_L, r_O, r_S) = \\ &\frac{1}{(2\pi)^3}\int dr'_L \; dr'_O \int d\beta_L \; d\beta_O \; g\left(\hat{\beta}\cdot(\vec{r'}-\gamma(\vec{r}-t\vec{\beta}))\right)\frac{\gamma^3}{\beta}S_{\vec{\beta}}(r'_L, r'_O, r'_S).\end{aligned} \tag{13}$$

Here $g(z) = \int_{-\infty}^{\infty} d\mu \; \mu^2 e^{-i\mu z}$ is a universal filter function. Due to our choice of coordinates, $r_S = r'_S$. The derivation of the inverse in (13) is straightforward following the appendix of Ref. [16], so we postpone the detailed derivation for a future article [20].

Let us now comment on inverting Eq. (9) in practice. We might just plug the imaged source into Eq. (13) and integrate to find the relative distribution of space-time points. This method is called the filtered back-projection algorithm and it is a poor way to perform the inversion in practice because of the singular behavior of the filter function g. A regularization of the filter function is possible, but introduces numerical inaccuracy [18]. Furthermore, the filtered back-projection algorithm is computationally intensive owing to the four integrals in Eq. (13).

A better approach is to use Algebraic Reconstructive Tomography (ART) [18,19]. In ART, one expands both the imaged source function and the relative distribution of space-time emission points in Eq. (9) in some function basis. For the sake of illustration we will use Basis splines [21] for this basis. In this basis, $S_{\vec{P}}(\vec{r'}) = \sum_{ij} S_{ij}B_i(\vec{P})B_j(\vec{r'})$ and $\tilde{\mathcal{S}}(r) = \sum_k \tilde{\mathcal{S}}_k B_k(r)$ where i, j, and k are indices in the directions of $\vec{P}, \vec{r'}$, and r respectively. Eq. (9) may then be recast as

$$\sum_{ij} S_{ij}B_i(\vec{P})B_j(\vec{r'}) = \int dt' \; \sum_k \tilde{\mathcal{S}}_k B_k(r). \tag{14}$$

Since the Basis splines are orthogonal, one can write this as a matrix equation:

$$S_{ij} = \sum_k \tilde{\mathcal{S}}_k \int d^3P \;\; d^3r' \; B_i(\vec{P})B_j(\vec{r'}) \int dt' \; B_k(r) \tag{15}$$

Only the t' integral needs to be done numerically as the other six integrals may be done analytically. Thus, ART reduces to inverting a matrix equation for the coefficients of the Basis spline expansion. We will explore the viability of the this approach in a future article [20].

In conclusion, we should be able to reconstruct the relative distribution of emission space and time points from the correlation functions measured in heavy-ion collisions. Our approach is model-independent and works for any like-pair correlations as it is implemented at the level of source functions. The momentum averaging approximation involved in the reconstruction seems to be justified in some experiments and its validity can be checked experimentally. Thus, this reconstruction provides us with a way to directly measure the freeze-out duration of heavy-ion collisions.

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